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LETTER TO THE EDITOR

On the extended characterisation theorem for quantum logics

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Abstract. An inaccuracy in the definition of dual automorphism of a lattice by Sharma and Mukherjee is corrected and some interesting ramifications are discussed.

Sharma and Mukherjee (1977) have proved an extended characterisation theorem for quantum logics. The purpose of this work is to draw attention to and correct an interesting inaccuracy in their definition of dual automorphism of a lattice. According to Sharma and Mukherjee (1977) an order reversing bijection from a lattice to itself is called a dual automorphism of the lattice.

First we give a counterexample to show that an order reversing bijection on a lattice need not have an order reversing inverse. Consider the lattice $Z^2 = Z \times Z$ ordered by inheriting the natural order of Z, that is

$$(l, m) \leqslant (r, s)$$

if and only if $l \leqslant r$ and $m \leqslant s$. It is evident that with this ordering Z^2 is a lattice. Consider the map with the matrix

$$\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}.$$

It is order reversing as all its entries are negative. It is also invertible, but its inverse has matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$$

which is neither order reversing nor order preserving.

It is clear that to be a dual automorphism, a map from a lattice to itself should be a lattice dual morphism (that is, one which takes joins into meets and meets into joins) and should be such that it has an inverse which also is a lattice dual morphism. However, by a simple adaptation of an easily proved result stated in Birkhoff (1973, p 134), we can prove that a bijective lattice dual morphism from a lattice to itself has an inverse which is also a lattice dual morphism and hence a dual automorphism. An order reversing map between lattices is called a dual morphism of order (MacLane and Birkhoff 1967, p 483); our counterexample, therefore, shows that even an invertible dual morphism of order is not necessarily a lattice dual morphism. On the other hand, by a simple adaptation of another result given in Birkhoff (1973, p 24), it is easy to prove that an order reversing bijection from a lattice to itself with an order reversing inverse is a dual automorphism of the lattice. Hence the following two definitions of dual automorphisms are equivalent and correct.

Definition 1. An order reversing bijection from a lattice to itself with an order reversing inverse is called a dual automorphism of the lattice.

Definition 2. A bijective lattice dual morphism from a lattice to itself is called a dual automorphism of the lattice.

All that has been said above with simple adaptations becomes relevant for the definitions of lattice isomorphisms and automorphisms also.

We finally note in passing that the good algebraic theorem (Birkhoff 1973, p 134) that a bijective morphism is an isomorphism does not apply to morphisms of order nor to dual morphisms of order!

References

Birkhoff G 1973 *Lattice Theory* (Providence, RI: American Mathematical Society)
MacLane S and Birkhoff G 1967 *Algebra* (New York: Macmillan)
Sharma C S and Mukherjee M K 1977 *J. Phys. A: Math. Gen.* **10** 1665–71